

IDEALS GENERATED BY 4 QUADRIC POLYNOMIALS

CRAIG HUNEKE, PAOLO MANTERO, JASON MCCULLOUGH,
AND ALEXANDRA SECELEANU

ABSTRACT. Motivated by Stillman's question, we show that the projective dimension of an ideal generated by four quadric forms in a polynomial ring has projective dimension at most 9.

1. INTRODUCTION

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k , and let $I = (f_1, \dots, f_N)$ be a homogeneous ideal of S . Classical theorems of Hilbert or Auslander and Buchsbaum give upper bounds on the projective dimension of S/I (or, equivalently, of I) in terms of n , the number of variables of S . Motivated by computational efficiency issues, Stillman [27] posed the following question:

Question 1.1 (Stillman [27, Problem 15.8]). *Is there a bound on $\text{pd}(S/I)$ depending only on d_1, \dots, d_N , and N , where $d_i = \deg(f_i)$?*

While this question is still open, several partial results are known:

- (1) When I is generated by N quadric forms, Ananyan-Hochster [1] showed that $\text{pd}(S/I)$ has an upper bound asymptotic to $2N^{2N}$. They also give more general result for non-homogeneous ideals generated in degree at most 2.
- (2) More recently, in July 2013 at a conference “Commutative algebra and its interactions with algebraic geometry” at the Centre International de Recontres Mathématiques, Ananyan-Hochster have announced a positive answer to Stillman's question for ideals generated in degree at most 4 if $\text{char}(k) \neq 2, 3$.
- (3) When I is minimally generated by N quadrics and $\text{ht}(I) = 2$, we previously showed [16] that $\text{pd}(S/I) \leq 2N - 2$. See Theorem 2.19.
- (4) When I is minimally generated by 3 cubics, Engheta [12] showed that $\text{pd}(S/I) \leq 36$, while the largest known example satisfies $\text{pd}(S/I) = 5$.

If Question 1.1 has an affirmative answer, any bound will necessarily be very large. Examples of Beder et. al. [2] show that projective dimension of ideals of 3 degree- d forms can grow exponentially with respect to d .

Further motivating Question 1.1, Caviglia showed it was equivalent to the following analogous question for Castelnuovo-Mumford regularity (cf. [24, Theorem 2.4]).

Question 1.2 (Stillman [27, Problem 15.8]). *Let $S = k[x_1, \dots, x_n]$, and I be an ideal generated by N homogeneous polynomials of given degrees d_1, \dots, d_N , and N . Is there a bound on $\text{reg}(S/I)$ only depending on d_1, \dots, d_N ?*

While (1) and (2) above show that $\text{pd}(S/I)$ is bounded for any ideal generated by N quadrics, these bounds are exponential in N . It is clear already in the case of an ideal generated by 3 quadrics that these bounds are far from optimal (cf. [24, Proposition 24]). In this paper we give a near optimal upper bound on the projective dimension of an ideal generated by 4 quadrics. Specifically, we prove

Theorem 9.1. *Let S be a polynomial ring over a field k , and let $I = (q_1, q_2, q_3, q_4)$ be an ideal of S generated by 4 homogeneous polynomials of degree 2. Then $\text{pd}(S/I) \leq 9$.*

Examples from [23] of ideals I generated by 4 quadrics with $\text{pd}(S/I) = 6$ show that the above bound is close to optimal. Our proof follows a technique similar to Engheta's in [9] and [12] in dividing into cases by height and multiplicity. We notably rely on the tight bound for the height 2 case proved in [16] and the height 3 multiplicity 6 case proved in [18]. The remaining height 3 cases constitute the bulk of this paper and require a variety of approaches.

The rest of the paper is organized as follows: in Section 2, we collect many of the results that we cite and fix notation; in Section 3, we prove some simple results that are used several times throughout the paper; in Section 4, we prove results on the structure of small matrices of linear forms; in Sections 5, 6, 7, and 8, we handle the four major cases for the proof of Theorem 9.1. In particular, Section 6 contains structure theorems for ideals primary to primes of height 3 and multiplicity 2 that may be of independent interest. Finally, we collect all of the previous results to prove our main result in Section 9.

2. BACKGROUND AND NOTATION

In this section we set the notation that will be used throughout this manuscript and collect several results that will be used in the following sections.

The first result is a well-known analog of Cramer's rule in Linear Algebra.

Lemma 2.1. *Let S be a commutative ring, \mathbf{A} an $m \times n$ matrix of elements of S , and x_1, \dots, x_n elements of S . If*

$$\mathbf{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0,$$

then $I_n(\mathbf{A}) \subseteq 0 : J$, where $J = (x_1, \dots, x_n)$ and $I_n(\mathbf{A})$ is the ideal generated by the $n \times n$ minors of \mathbf{A} .

2.1. Unmixed ideals and multiplicity. We will frequently use the following well-known *associativity formula* (also sometimes referred to as the *linearity formula* or *additivity and reduction formula*) to compute the multiplicity of an ideal.

Proposition 2.2 (cf. [31, Theorem 11.2.4]). *If J is an ideal of S , then*

$$e(S/J) = \sum_{\substack{\text{primes } \mathfrak{p} \supseteq J \\ \text{ht}(\mathfrak{p}) = \text{ht}(J)}} e(S/\mathfrak{p}) \lambda(S_{\mathfrak{p}}/J_{\mathfrak{p}}).$$

Here $e(M)$ denotes the Hilbert-Samuel multiplicity of an S -module M with respect to the graded maximal ideal, and $\lambda(M)$ denotes the length of M .

Recall that an ideal J of height h is *unmixed* if $\text{ht}(\mathfrak{p}) = h$ for every $\mathfrak{p} \in \text{Ass}(S/J)$. Also, the *unmixed part* of J , denoted J^{un} , is the intersection of all the components of J of minimum height $\text{ht}(J)$. By definition one has $J \subseteq J^{un}$, and from the Associativity Formula it follows that $e(S/J) = e(S/J^{un})$. We follow [9] in using the following notation: If J is an unmixed ideal, we say it is of type

$$\langle e_1, \dots, e_m; \lambda_1, \dots, \lambda_m \rangle$$

if J has m associated prime ideals of minimal height $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ with $e_i = e(S/\mathfrak{p}_i)$ and $\lambda(S_{\mathfrak{p}_i}/J_{\mathfrak{p}_i}) = \lambda_i$ for all i . It follows from the associativity formula that $e(S/J) = \sum_{i=1}^m e_i \lambda_i$.

The following simple fact also follows from the associativity formula:

Lemma 2.3 (Engheta cf. [9, Lemma 8]). *Let $J \subset S$ be an unmixed ideal. If $I \subset S$ is an ideal containing J such that $\text{ht}(I) = \text{ht}(J)$ and $e(S/I) = e(S/J)$, then $J = I$.*

Engheta's finite characterization of height 2 unmixed ideals of multiplicity 2 gives a bound on projective dimension in this case.

Proposition 2.4 ([9, Proposition 11]). *Let I be an unmixed ideal in a polynomial ring S over an algebraically closed field. Suppose $\text{ht}(I) = 2$ and $e(S/I) = 2$. Then $\text{pd}(S/I) \leq 3$.*

In [17] it was shown that this characterization is very special and cannot be extended to higher heights or multiplicities. In fact, for any multiplicity $e \geq 2$ and any height $h \geq 2$ with $(e, h) \neq (2, 2)$ there are infinitely many distinct unmixed ideals of multiplicity e and height h . Moreover, they can be chosen to have arbitrarily large projective dimension.

2.2. Linkage. Two ideals J and K in a regular ring R are *linked*, denoted $J \sim K$, if there exists a regular sequence $\alpha = \alpha_1, \dots, \alpha_g$ such that $K = (\alpha) : J$ and $J = (\alpha) : K$ (notice that the definition forces J and K to be unmixed, and $\alpha \subseteq J \cap K$). Linkage has been studied since the nineteenth century, although its first modern treatment appeared in the ground-breaking paper

by Peskine and Szpiro [27]. We refer the interested reader to [25] and [19] and their references.

We will need a few results from linkage. The original results were stated for local Gorenstein rings but we state the graded versions over the polynomial ring S .

Proposition 2.5. (Peskine-Szpiro, [27]) *Let J be an unmixed ideal of S of height g . Let $\alpha = \alpha_1, \dots, \alpha_g$ be a regular sequence in J , and set $K = (\alpha) : J$. Then, one has $J = (\alpha) : K$, that is, $J \sim K$ via α .*

It is easily checked that if α is a regular sequence of maximal length in an ideal J , then $(\alpha) : J = (\alpha) : J^{un}$, that is, J^{un} is linked to $(\alpha) : J$. We will use this fact several times.

The following fact is well-known.

Lemma 2.6 (cf. [9, Theorem 7]). *Let J be an almost complete intersection ideal of S . If K is any ideal linked to J^{un} , then one has*

$$\text{pd}(S/J) \leq \text{pd}(S/K) + 1.$$

The following result collects a couple of well-known properties of linked ideals.

Theorem 2.7. (Peskine-Szpiro, [27]) *Let J and K be homogeneous ideals of S that are linked via a regular sequence α . Then, one has*

- (1) *S/J is Cohen-Macaulay if and only if S/K is Cohen-Macaulay;*
- (2) *$e(S/J) + e(S/K) = e(S/(\alpha))$.*

The next lemma is an ideal-theoretic version of a classical result of Samuel (in the equicharacteristic case) and Nagata (in full generality): in our setting, it states that any unmixed homogeneous S -ideal of multiplicity 1 is a prime ideal generated by linear forms (sometimes this is called a *regular prime*).

Theorem 2.8. (Samuel [28], Nagata [26, Theorem 40.6]) *Let J be a homogeneous, unmixed ideal of S . If $e(S/J) = 1$, then J is generated by part of a homogeneous regular system of parameters of S .*

The following is a special case of a result of Huneke on residual intersections.

Proposition 2.9 (Huneke [14, Theorem 3.1]). *Let S be a polynomial ring. Let C be a complete intersection homogeneous ideal in S . Let $A = (a_1, \dots, a_s)$ denote an ideal contained in C . Set $J = A : C$ and assume $\text{ht } J \geq s$ and $\text{ht}(C + J) \geq s + 1$. Then $A = J \cap C$.*

2.3. 1-generic matrices. The class of 1-generic matrices was introduced in Eisenbud's paper [5]. We explain in Section 4 how matrices of linear forms come up naturally when analyzing ideals generated by quadrics that are contained in a linear prime.

Let M be a matrix of linear forms over a polynomial ring S with residue field k . By a *generalized row* of M we mean a k -linear combination of the

rows of M with not all coefficients 0. Similarly a *generalized column* is a non-zero k -linear combination of the columns of M . A *generalized entry* of M is simply a linear combination with non-zero coefficients of the entries of some generalized row. In the following we write $I_i(M)$ for the ideal of $i \times i$ minors of M .

Definition 2.10. The matrix M is called *1-generic* if, after arbitrary k -linear row and column operations, M exhibits no generalized zero entries.

The following result was established in [6] (see also Theorem 2.1 in [5] for a generalization allowing linear sections of small codimension of these determinantal varieties).

Theorem 2.11 (Eisenbud, [6, Theorem 6.4]). *If M is a 1-generic matrix of linear forms of size $p \times q$ ($p \leq q$) with entries in a polynomial ring S over an algebraically closed field k , then the ideal $I_p(M)$ generated by the maximal minors of M is prime of codimension $q - p + 1$. Its free resolution is given by an Eagon-Northcott complex and $S/I_p(M)$ is a Cohen-Macaulay domain.*

In addition to using the preceding theorem for 1-generic matrices, we shall also be concerned with ideals of minors of matrices which are far from being 1-generic; specifically we shall be interested in determining the height of ideals of minors of matrices whose rank is not maximal. In this situation, bounds on the height of ideals of minors for (not necessarily linear) matrices have been given by Eisenbud, Huneke and Ulrich [8] generalizing results of Eagon-Northcott, Bruns and Faltings.

Theorem 2.12 (Eisenbud-Huneke-Ulrich, [8, Theorem A]). *Let R be a regular local ring, and let M be a matrix of size $p \times q$ with entries in R . Set $r = \text{rank}(M)$ and consider an integer i such that $1 \leq i \leq r$ and $I_i(M) \neq R$. Then*

$$\text{ht } I_i(M) \leq (r - i + 1)(\max\{p, q\} - i + 1) + i - 1.$$

Remark 2.13. The conclusion of the theorem above holds in the graded case as well. If S is a polynomial ring and M a matrix of size $p \times q$ and rank r whose entries are homogeneous forms, then

$$\text{ht } I_i(M) \leq (r - i + 1)(\max\{p, q\} - i + 1) + i - 1.$$

This is easily deduced from Theorem 2.12 by localizing at the homogeneous maximal ideal.

2.4. Height three primes of small multiplicity. Here we recall some of the structure theorems regarding prime ideals of small height and multiplicity that we shall require. For most of this section we need the field k to be algebraically closed. The reduction to this case occurs in the proof of Theorem 9.1. Also, recall that a homogeneous ideal J is called *degenerate* if J contains at least one linear form, otherwise J is said to be *non-degenerate*. The first result we need is a classical simple lower bound for the multiplicity of non-degenerate prime ideals.

Proposition 2.14 ([13, Corollary 18.12]). *Let \mathfrak{p} be a homogeneous prime ideal of S . If \mathfrak{p} is a non-degenerate prime ideal, then $e(S/\mathfrak{p}) \geq \text{ht}(\mathfrak{p}) + 1$.*

The following result is a trivial consequence of Proposition 2.14.

Corollary 2.15. *Let \mathfrak{p} be a homogeneous prime ideal of S of height three. If $e(S/\mathfrak{p}) = 2$, then there exist linear forms x, y and a quadric q such that $\mathfrak{p} = (x, y, q)$.*

The next three results are classification theorems of varieties of small degree over an algebraically closed field. We give an equation-oriented version of these results that best fits our purpose. The first of these results follows from a classical classification theorem for varieties of multiplicity 3. (See [29].) By Proposition 2.14 all such primes are degenerate.

Theorem 2.16 (Anonymous [30], Swinnerton-Dyer [29, Theorem 3]). *Let \mathfrak{p} be a homogeneous prime ideal of S of height three. If $e(S/\mathfrak{p}) = 3$, then \mathfrak{p} is of one of the following types:*

- (1) $\mathfrak{p} = (x, y, c)$ where x and y are linear forms and c is a cubic form;
- (2) $\mathfrak{p} = (x) + I_2(\mathbf{M})$, where \mathbf{M} is a 2×3 matrix of linear forms.

The next result is a classification theorem for varieties of multiplicity 4. The non-degenerate case was done by Swinnerton-Dyer [29, Theorem 1]; the degenerate cases are derived from work of Brodmann-Schenzel [3, Theorem 2.1]).

Theorem 2.17. *Let \mathfrak{p} a homogeneous prime ideal of S of height 3. If $e(S/\mathfrak{p}) = 4$, then \mathfrak{p} is of one of the following types:*

- (1) $\mathfrak{p} = (x, y, r)$, where x, y are linear forms and r is a quartic,
- (2) $\mathfrak{p} = (x, q, q')$, where x is a linear form and q and q' are quadrics,
- (3) $\mathfrak{p} = I_2(\mathbf{M})$, where \mathbf{M} is a 2×4 matrix of linear forms or a 3×3 skew-symmetric matrix of linear forms.
- (4) \mathfrak{p} is obtained via a finite sequence of hyperplane sections of one of the ideals of part (3),
- (5) either \mathfrak{p} is generated by a linear form, a quadric and three cubics, or \mathfrak{p} is generated by a linear form and 7 cubics.

The following result only describes the Betti tables of the non-degenerate homogeneous primes of multiplicity five. Since the classification of height two homogeneous primes \mathfrak{p} with $e(S/\mathfrak{p}) = 5$ is not known, we do not know the precise structure of a degenerate height 3 prime ideal \mathfrak{p} with $e(S/\mathfrak{p}) = 5$. However, we will see soon that, for our purposes, it suffices to know the structure in the non-degenerate situation.

Theorem 2.18. (Brodmann-Schenzel, [3, Theorem 2.2]) *Let \mathfrak{p} be a non-degenerate homogeneous prime ideal of height 3. If $e(S/\mathfrak{p}) = 5$, then \mathfrak{p} satisfies one of the following:*

- (1) \mathfrak{p} is a Gorenstein ideal generated by the 4 by 4 Pfaffians of a 5 by 5 skew-symmetric matrix of linear forms,

(2) S/\mathfrak{p} has Betti table:

	0	1	2	3	4
0:	1	-	-	-	-
1:	-	4	2	-	-
2:	-	1	6	5	1

(3) S/\mathfrak{p} has Betti table:

	0	1	2	3	4	5
0:	1	-	-	-	-	-
1:	-	3	2	-	-	-
2:	-	6	16	15	6	1

2.5. Related results. Finally we recall two results of the authors' needed for the main result:

Theorem 2.19 ([16, Theorem 3.5]). *For any ideal I of height two generated by n quadrics in a polynomial ring S , one has $\mathrm{pd}(S/I) \leq 2n - 2$. Moreover, this bound is tight.*

In particular, an ideal generated by 4 quadrics of height two satisfies $\mathrm{pd}(S/I) \leq 6$.

The following is a generalization of a result of Engheta.

Theorem 2.20 ([18, Corollary 2.8], [10, Theorem 1]). *Let I be an almost complete intersection of height $g > 1$ generated by quadrics. Then $e(S/I) \leq 2^g - g + 1$. Moreover, if $e(S/I) = 2^g - g + 1$, then S/I is Cohen-Macaulay.*

Engheta proved an upper bound in the case of any graded almost complete intersection. The Cohen-Macaulayness and a more general bound are given in [18]. In particular, this result shows that an ideal I of height 3 generated by 4 quadrics has multiplicity at most 6, and moreover, if I has multiplicity exactly 6, it is Cohen-Macaulay and hence $\mathrm{pd}(S/I) = 3$.

Finally, we make use of the following observation of Ananyan-Hochster:

Lemma 2.21 ([1, Lemma 3.3]). *Let f_1, \dots, f_t be a regular sequence of forms in S . Then S is free, hence, faithfully flat over $A = k[f_1, \dots, f_t]$. Hence, for any ideal I of S whose generators lie in A , $\mathrm{pd}(S/I) \leq t$.*

3. BASIC RESULTS

In this section we collect some basic results and notation. If I is an ideal generated by 4 quadric forms in a polynomial ring S , then $\mathrm{ht}(I) \leq 4$. If $\mathrm{ht}(I) = 1$ or 4, then $\mathrm{pd}(S/I) = 4$. We will see in the proof of Theorem 9.1 that it suffices to consider the height three case.

Assumptions 3.1. *Unless otherwise specified, we use the following notation in the remainder of the paper:*

- S is a polynomial ring over an algebraically closed field k ,
- $I = (q_1, \dots, q_4)$ is an S -ideal of height 3,

- q_1, \dots, q_4 are homogeneous polynomials of degree 2,
- $\text{ht}(q_1, q_2, q_3) = 3$.

Next, we remark that if $e(S/I) = 1$, then $\text{pd}(S/I) \leq 4$.

Proposition 3.2. *If $e(S/I) = 1$, then $\text{pd}(S/I) \leq 4$.*

Proof. Let J be any ideal linked to I^{un} . By Theorem 2.8, I^{un} is Cohen-Macaulay, and by Theorem 2.7, J is also Cohen-Macaulay. Lemma 2.6 now implies $\text{pd}(S/I) \leq \text{pd}(S/J) + 1 = 4$. \square

Lemma 3.3. *Assume I^{un} contains a linear form. Then $\text{pd}(S/I) \leq 5$.*

Proof. Let x be a linear form contained in I^{un} . Since $I \subseteq (I, x) \subseteq I^{un}$ we have $\text{ht}(I, x) = 3$. Hence, after possibly relabeling the quadrics q_1, \dots, q_4 , we may assume x, q_1, q_2 form a regular sequence. Let $J = (x, q_1, q_2) : I = (x, q_1, q_2) : I^{un}$. By Theorem 2.5 one has $e(S/J) + e(S/I^{un}) = e(S/(x, q_1, q_2)) = 4$, whence one obtains $e(S/J) \leq 3$. If $e(S/J) = 1$ then by Theorem 3.2 one has S/J is Cohen-Macaulay. If $e(S/J) = 3$ then $e(I^{un}) = 1$, so again S/J is Cohen-Macaulay by Theorems 2.7 and 3.2. In either case $\text{pd}(S/J) = 3$, and by lemma 2.6, this gives $\text{pd}(S/I) \leq 4$.

Finally if $e(S/J) = 2$, one can write $J = (J', x)$ for an ideal J' of height two, whose minimal generators do not involve the variable x (in particular, x is regular on S/J'). The ideal J' is unmixed of height 2 and multiplicity 2, hence by Proposition 2.4, one has $\text{pd}_S(S/J') \leq 3$. This yields $\text{pd}(S/J) \leq 4$ (because x is regular on S/J') which, by Lemma 2.6, gives $\text{pd}(S/I) \leq 5$. \square

Lemma 3.4. *If I^{un} is Cohen-Macaulay, then $\text{pd}(S/I) \leq 4$.*

Proof. We may suppose that $I = (q_1, q_2, q_3, q_4)$ and $C = (q_1, q_2, q_3)$ is a complete intersection. By Proposition 2.7, $C : I = C : I^{un}$ is Cohen-Macaulay. By Lemma 2.6, $\text{pd}(S/I) \leq 4$. \square

Lemma 3.5. *Suppose $I \subseteq L \cap J$, where $L = (L', q)$, $\text{ht}(L) = \text{ht}(J) = 3$, $\text{ht}(L') = 2$ and q is a quadric. Then there exists a quadric $q' \in L \cap J$ such that $L = (L', q')$.*

Proof. For each $i = 1, \dots, 4$, write $q_i = f_i + \alpha_i q$, where f_i is an element of L' and $\alpha_i \in k$. If $\alpha_i = 0$ for all i , one has $I \subseteq L'$, which implies $\text{ht} L' \leq 2$ and gives a contradiction. Hence, one may assume $\alpha_i \neq 0$ for some i . Now, take $q' = q_i$. \square

Lemma 3.6. *Suppose I_1, I_2 are ideals and $J = I_1 \cap I_2$. Further suppose $I_1 = (f) + J'$, where $J' \subset I_2$ and f is regular on S/I_2 . Then*

$$\text{pd}(S/J) \leq \max\{\text{pd}(S/I_1), \text{pd}(S/I_2)\}.$$

In particular, if both S/I_1 and S/I_2 are Cohen-Macaulay, so is S/J .

Proof. The hypotheses imply that $J + (f) = I_1$ and $J : (f) = I_2$. The statement then follows from the following short exact sequence

$$0 \rightarrow \frac{S}{J : (f)} \xrightarrow{\cdot f} \frac{S}{J} \rightarrow \frac{S}{J + (f)} \rightarrow 0.$$

□

Lemma 3.7. *Let F_\bullet be a free resolution of I^{un} and let $d_i : F_i \rightarrow F_{i-1}$ denote the i th differential. Then $\text{pd}(S/I) \leq \max\{5, \text{pd}(\text{Ker}(d_4^*)) + 2\}$, where $d_4^* = \text{Hom}_S(d_4, S)$.*

Proof. Let $C = (q_1, q_2, q_3)$ be a height 3 complete intersection in $I = (q_1, q_2, q_3, q_4)$ and set $L = C : I = C : I^{un}$. By [17, Lemma 3.3], $\text{pd}(\text{Im}(d_3^*)) = 2$. From the short exact sequence

$$0 \rightarrow \text{Im}(d_3^*) \rightarrow \text{Ker}(d_4^*) \rightarrow \text{Ext}_S^3(S/I^{un}, S) \rightarrow 0,$$

we have

$$\text{pd}(\text{Ext}_S^3(S/P, S)) \leq \max\{\text{pd}(\text{Im}(d_3^*)) + 1, \text{pd}(\text{Ker}(d_4^*))\} = \max\{3, \text{pd}(\text{Ker}(d_4^*))\}.$$

By [17, Lemma 3.1], $\frac{C:I}{C} \simeq \text{Ext}_S^3(S/I^{un}, S)$. From the short exact sequence

$$0 \rightarrow \frac{C:I}{C} \rightarrow \frac{S}{C} \rightarrow \frac{S}{L} \rightarrow 0,$$

we have

$$\text{pd}(S/L) \leq \max\{\text{pd}(S/C), \text{pd}((C:I)/C) + 1\} = \max\{4, \text{pd}(\text{Ker}(d_4^*)) + 1\}.$$

By Lemma 2.6, we have $\text{pd}(S/I) \leq \max\{5, \text{pd}(\text{Ker}(d_4^*)) + 2\}$. □

4. MATRICES OF LINEAR FORMS

In this section we prove some structure theorems for matrices of linear forms over an arbitrary polynomial ring S over a field k .

The following lemma is written in a form that will make it handy for applications to larger matrices. We will need it in the proof of Lemma 4.2.

Lemma 4.1. *Let $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2 by 2 matrix whose entries are linear forms in S . If $\det \mathbf{M} = 0$, then one of the following holds:*

- (1) *Two entries in a row or column are 0.*
- (2) *There exists $0 \neq \alpha \in k$ such that*

$$(i) \mathbf{M} = \begin{pmatrix} a & b \\ \alpha a & \alpha b \end{pmatrix}, \text{ or}$$

$$(ii) \mathbf{M} = \begin{pmatrix} a & \alpha a \\ c & \alpha c \end{pmatrix}.$$

Proof. By assumption, one has $ad = bc$. Since a is a prime element, either $b \in (a)$ or $c \in (a)$. If $a = 0$, then either $b = 0$ or $c = 0$ and the statement follows. If $a \neq 0$, then either $b = \alpha a$ or $c = \alpha a$, for some $\alpha \in k$. In the case $b = \alpha a$, from $ad = bc$ one obtains $d = \alpha c$, which gives the desired statement. Similarly, $c = \alpha a$ implies $d = \alpha b$. □

The next result gives the precise forms of a 3 by 3 matrix of linear forms whose determinant is zero. It is one of the two crucial steps to prove Theorem 5.1.

Lemma 4.2. *Let \mathbf{M} be a 3 by 3 matrix with entries that are linear forms in S . If $\det \mathbf{M} = 0$, then there exists invertible 3 by 3 matrices \mathbf{E} and \mathbf{F} such that \mathbf{EMF} has one of the following shapes*

$$(i) \begin{pmatrix} \cdot & \cdot & 0 \\ \cdot & \cdot & 0 \\ \cdot & \cdot & 0 \end{pmatrix}, (ii) \begin{pmatrix} 0 & 0 & \cdot \\ 0 & 0 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, (iii) \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & 0 \end{pmatrix}, (iv) \begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix},$$

where a, b, c are linear forms and \cdot indicates an arbitrary linear form.

Proof. By a classical result of Eisenbud [5, Theorem 2.1], the matrix \mathbf{M} has a generalized zero, that is, after elementary row and column operations one may assume that one of the entries of \mathbf{M} is 0. Without loss of generality, we may assume \mathbf{M} has the following shape:

$$\mathbf{M} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & 0 \end{pmatrix} \quad (*)$$

where a, b, \dots, h are linear forms.

If $\text{ht}(c, f) = 0$, then $c = f = 0$ and we are in case (i); hence, we may assume $\text{ht}(c, f) \geq 1$.

If $\text{ht}(c, f) = 1$, after an elementary row operation we may assume $c \neq 0$ and $f = 0$. Since $\det \mathbf{M} = 0$ and c is a non zero divisor, we obtain $\det \begin{pmatrix} d & e \\ g & h \end{pmatrix} = 0$. By Lemma 4.1, one can perform elementary row or column operations in \mathbf{M} to obtain two zeroes in the same row or column in the submatrix $\begin{pmatrix} d & e \\ g & h \end{pmatrix}$ of \mathbf{M} . If the two zeroes are on the same row we are in case (iii). If the two zeroes are in the same column, after permuting rows in \mathbf{M} we obtain case (ii).

Finally, consider the case $\text{ht}(c, f) = 2$. If $\text{ht}(d, e, f) = 1$, then after adding appropriate multiples of the last column to the first two columns, we may assume \mathbf{M} has the form

$$\begin{pmatrix} a & b & c \\ 0 & 0 & f \\ g & h & 0 \end{pmatrix},$$

and $f \neq 0$. Since the determinant still vanishes, we have $\det \begin{pmatrix} a & b \\ g & h \end{pmatrix} = 0$.

Applying Lemma 4.1 one can easily obtain either a block $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ in the matrix or a column of zeroes.

Hence, we may further assume $\text{ht}(d, e, f) \geq 2$. Consider the case $\text{ht}(d, e, f) = 3$. The Laplace expansion formula to compute the determinant of \mathbf{M} in (*)

with respect to the last column gives that $c \cdot \det \begin{pmatrix} d & e \\ g & h \end{pmatrix} \in (f)$. Since c is

a regular element on $\overline{S} = S/(f)$, one deduces that the image of $\det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$

is 0 in $\overline{S} = S/(f)$. Now, Lemma 4.1 together with $\text{ht}(d, e, f) = 3$ imply that there exist elements α, β, γ in k such that \mathbf{M} has the form

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ \alpha d + \beta f & \alpha e + \gamma f & 0 \end{pmatrix}.$$

If $\alpha \neq 0$, adding α^{-1} times the last row to the second row returns us to the case where $\text{ht}(d, e, f) = 1$; hence, we may assume $\alpha = 0$. In this situation, after possibly adding a proper multiple of one of the first two columns to the other one (and permuting the first two columns), we may assume $\gamma = 0$, that is, we are in the situation where \mathbf{M} has the form

$$\begin{pmatrix} a & b & c \\ d & e & f \\ \beta f & 0 & 0 \end{pmatrix}.$$

Computing the determinant, we obtain $\beta f \cdot \det \begin{pmatrix} b & c \\ e & f \end{pmatrix} = 0$. Since f is regular, either $\beta = 0$, which yields case (iii), or $\beta \neq 0$ and $\det \begin{pmatrix} b & c \\ e & f \end{pmatrix} = 0$. In this latter case, Lemma 4.1 together with $\text{ht}(c, f) = 2$, gives the equality of matrices

$$\begin{pmatrix} b & c \\ e & f \end{pmatrix} = \begin{pmatrix} \alpha c & c \\ \alpha f & f \end{pmatrix}$$

for some $\alpha \in k$. Subtracting α times the second column from the first column, we obtain a column of zeroes, as in case (i).

Therefore, we only need to consider the case where $\text{ht}(d, e, f) = 2$. If $\text{ht}(d, f) = 1$, then by adding a multiple of the third column to the first, we may assume $d = 0$. If $\text{ht}(d, f) = 2$, then since (d, f) is prime, $e \in (d, f)$. After adding multiples of the first and last columns to the second column, we may assume $e = 0$. In either case, we may permute the columns if necessary and now assume \mathbf{M} has the form

$$\begin{pmatrix} a & b & c \\ d & 0 & f \\ g & h & 0 \end{pmatrix}.$$

As the determinant is 0, we have $cdh \in (f)$, which in turn implies $h \in (f)$ (because c and d are regular on $S/(f)$ by assumption). If $h = 0$, then $\det \mathbf{M} = gb f = 0$, whence either $g = 0$ or $b = 0$. In the first case we obtain case (iii); in the second case, case (i). Hence, we may assume $h \neq 0$. Without loss of generality, we may assume $h = f$. So we have the matrix

$$\begin{pmatrix} a & b & c \\ d & 0 & f \\ g & f & 0 \end{pmatrix}.$$

Observe that if $\text{ht}(g, f) = 1$, an argument similar to the one used when $\text{ht}(c, f) = 1$ gives the desired statement. We may then assume $\text{ht}(g, f) = 2$.

Notice that if $\text{ht}(g, d, f) = 2$, then $d \in (g, f)$. After adding proper multiples of the last column to the first column, and of the last row to the second row, we may assume $d = 0$. The entry in position $(2, 2)$ does not need to be 0 anymore, but is a multiple of f . We are now back to the case where $\text{ht}(d, e, f) = 1$, hence the statement follows.

Therefore, we only need to consider the case where $\text{ht}(c, f) = \text{ht}(g, f) = 2$ and f is regular on $S/(g, d)$. Since the determinant is 0, we have $cdf = f(af - bg)$, whence $cd = af - bg$ and $af \in (d, g)$. By our assumption on f , this yields that $a \in (d, g)$. After adding to the first row appropriate multiples of the other two rows, we may assume $a = 0$. We can now assume \mathbf{M} has the form

$$\begin{pmatrix} 0 & b & c \\ d & 0 & f \\ g & f & 0 \end{pmatrix}.$$

Now, from $\det \mathbf{M} = 0$ we obtain $cd = -bg$. Since $\text{ht}(g, d) = 2$, this implies $c = \alpha g$ and $b = \alpha d$ for some $\alpha \in k$. Since $c \neq 0$, we have $\alpha \neq 0$. After multiplying the first row by α^{-1} we may assume $\alpha = 1$. Hence we have can assume \mathbf{M} has the form

$$\begin{pmatrix} 0 & -d & g \\ d & 0 & f \\ g & f & 0 \end{pmatrix}.$$

Multiplying the first two columns by -1 gives us case (iv) and completes the proof. \square

Using to Lemma 4.2, we can now give a similar structure result for 3×4 matrices.

Proposition 4.3. *Let \mathbf{M} be a 3×4 matrix whose entries are linear forms. If $\text{ht}(I_3(\mathbf{M})) = 0$, then either $\text{ht}(I_1(\mathbf{M})) \leq 6$ or there exist an invertible 3 by 3*

matrix \mathbf{E} and an invertible 4 by 4 matrix \mathbf{F} such that $\mathbf{EMF} = \begin{pmatrix} \cdot & \cdot & 0 \\ \cdot & \cdot & 0 \\ \cdot & \cdot & 0 \end{pmatrix}$.

Proof. Let \mathbf{M}_j denote the 3×3 minor of \mathbf{M} obtained by deleting the j th row. If one of the \mathbf{M}_j 's can be reduced to case (iv) of Lemma 4.2 via elementary column and row operations, then the statement follows immediately, because all the entries can be written in terms of at most 6 linear forms. Hence, we may assume that all the \mathbf{M}_j 's can be reduced to one of the cases (i), (ii), (iii) of Lemma 4.2. To prove the statement, it suffices to show that if there are no four zeroes in a column, then there are 6 generalized zeroes in \mathbf{M} .

We claim that, after row and column operations, we may assume that either \mathbf{M} contains three zeroes in a row or three zeroes in a column. If \mathbf{M}_4 is as in case (i) or (iii) in Lemma 4.2, this is clear. If \mathbf{M}_4 is as in case (ii),

we may assume

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & f \\ g & h & i \\ j & k & l \end{pmatrix}.$$

Since $\det(\mathbf{M}_1) = 0$, either $f = 0$, yielding three zeroes in a row, or $\det \begin{pmatrix} g & h \\ j & k \end{pmatrix} = 0$. By Lemma 4.1, another row or column operation will produce a third zero in one of the first two columns.

First we handle the case when \mathbf{M} has three zeroes in a row. Without loss of generality we may assume

$$\mathbf{M} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{pmatrix}.$$

An application of Lemma 4.2 to \mathbf{M}_4 gives at least another 3 generalized zeroes in \mathbf{M} , thus yielding $\text{ht}(I_1(\mathbf{M})) \leq 6$. Hence, we may assume that \mathbf{M} has three zeroes in a column. Then, without loss of generality, we can write

$$\mathbf{M} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 0 \\ g & h & i \end{pmatrix},$$

where $i \neq 0$. Since $\det(\mathbf{M}_1) = 0$ and $i \neq 0$, one has $\det \begin{pmatrix} c & d \\ e & f \end{pmatrix} = 0$. An application of Lemma 4.1 gives two new generalized zeroes in \mathbf{M}_1 that lie in the same row or column. If both lie in the same row, we have a row of zeroes and are done by the above argument. If the zeroes are in the same column, we may assume $c = e = 0$. Then we have $\det(\mathbf{M}_3) = adi = 0$, which, implies $a = 0$ or $d = 0$ and finishes the proof. \square

5. I^{un} HAS AN (x, y, z) -PRIMARY COMPONENT OF MULTIPLICITY $e \geq 2$

As a consequence of the structure theorems in the previous section, we can now prove that $\text{pd}(S/I) \leq 9$ holds in several cases.

Proposition 5.1. *Assume I is contained in an ideal P primary to a height 3 linear prime $\mathfrak{p} = (x, y, z)$ and with $e(S/P) \geq 2$. Then $\text{pd}(S/I) \leq 9$.*

Proof. Write $(q_1, \dots, q_4)^\top = \mathbf{M}(x, y, z)^\top + (f_1, \dots, f_4)^\top$, where $^\top$ denotes the transpose of a matrix, \mathbf{M} is a 3 by 4 matrix of linear forms, and f_1, \dots, f_4 are quadrics in $(x, y, z)^2$. Since $e(S/P) \geq 2$, we have $I_3(\mathbf{M})_{\mathfrak{p}} = 0$, which, by flatness of the map $S \rightarrow S_{\mathfrak{p}}$, implies $I_3(\mathbf{M}) = 0$.

We can now invoke Theorem 4.3, to obtain either $\text{ht}(I_1(\mathbf{M})) \leq 6$ or \mathbf{M} can be reduced via elementary column and row operations to have a column of zeroes. In the former case the four quadrics can be written in terms of at most 9 linear forms, whence $\text{pd}(S/I) \leq 9$.

Therefore, we only need to prove that $\text{pd}(S/I) \leq 9$ when \mathbf{M} can be reduced via elementary column and row operations to have a column of zeroes.

After possibly regrouping the terms, we may assume $q_i = a_i x + b_i y + \alpha_i z^2$ for every $i = 1, \dots, 4$, where the a_i and b_i are linear forms and $\alpha_1, \dots, \alpha_4 \in k$. We may also assume one of the α_i is non-zero, because, if not, we would have $I \subseteq (x, y)$, contradicting the fact that $\text{ht}(I) = 3$. Without loss of generality, we may assume $\alpha_1 = 1$. By adding appropriate multiples of q_1 to the other quadrics, we may assume $\alpha_i = 0$ for $i \geq 2$; that is, we can write $(q_1, \dots, q_4)^T = \mathbf{M}'(x, y)^T + (z^2, 0, 0, 0)^T$, where

$$\mathbf{M}' = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{pmatrix}.$$

Notice that if $\text{ht}(I_1(\mathbf{M}') + (x, y, z)) \leq 9$, then we can use at most 9 linear forms to write the 4 quadrics, giving $\text{pd}(S/I) \leq 9$. Hence, we may assume $\text{ht}(I_1(\mathbf{M}') + (x, y, z)) \geq 10$. Now, consider the matrices

$$\mathbf{A} = \begin{pmatrix} y & -x \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{pmatrix} \text{ and } \mathbf{A}' = \begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{pmatrix}.$$

Observe that $I_2(\mathbf{A}) = I_2(\mathbf{A}') + (q_2, q_3, q_4)$. Since we have $\mathbf{A}'(x, y)^T = (q_2, q_3, q_4)^T$, by Lemma 2.1, we obtain $I_2(\mathbf{A}')(x, y) \subseteq (q_2, q_3, q_4)$; that is, $I_2(\mathbf{A}') \subseteq J$, where $J = (q_2, q_3, q_4) : (x, y)$. Thus $I_2(\mathbf{A}) \subseteq J$.

First, notice that if \mathbf{A} is 1-generic, then by Theorem 2.11, $I_2(\mathbf{A})$ is a prime ideal of height 3. In particular, $\text{ht}(J) \geq 3$. Also, since $I_2(\mathbf{A})$ is a prime ideal that does not contain (x, y) , it is clear that $\text{ht}(I_2(\mathbf{A}) + (x, y)) \geq 4$; in particular also $\text{ht}(J + (x, y)) \geq 4$. By Proposition 2.9, we have $(q_2, q_3, q_4) = (x, y) \cap J$. From the chain of inclusions

$$(q_2, q_3, q_4) \subseteq (x, y) \cap I_2(\mathbf{A}) \subseteq (x, y) \cap J = (q_2, q_3, q_4)$$

we deduce $(q_2, q_3, q_4) = (x, y) \cap I_2(\mathbf{A})$. This fact together with $q_1 \notin (x, y)$ gives $(q_2, q_3, q_4) : q_1 = [I_2(\mathbf{A}) : q_1] \cap (x, y)$. Since $I_2(\mathbf{A})$ is prime, this intersection is either (x, y) (if $q_1 \in I_2(\mathbf{A})$) or (q_2, q_3, q_4) (if $q_1 \notin I_2(\mathbf{A})$). In either case (relying on Theorem 2.19 for the latter), $\text{pd}(S/(q_2, q_3, q_4) : q_1) \leq 4$. Consider the short exact sequence

$$0 \rightarrow S/((q_2, q_3, q_4) : q_1) \xrightarrow{q_1} S/(q_2, q_3, q_4) \rightarrow S/I \rightarrow 0.$$

Again by Theorem 2.19, $\text{pd}(S/(q_2, q_3, q_4)) \leq 4$, hence, for instance by the mapping cone construction, $\text{pd}(S/I) \leq 5$.

We may assume \mathbf{A} has precisely one generalized zero, say $b_4 = 0$, and $\text{ht}(I_1(\mathbf{M}') + (x, y, z)) = 10$; that is, all the non-zero entries of \mathbf{M} are linear forms that are linearly independent. In this case, $I = (a_1 x + b_1 y + z^2, a_2 x + b_1 y, a_3 x + b_3 y, a_4 x)$. Clearly $q_1 = a_1 x + b_1 y + z^2$ is regular on $S/(q_2, q_3, q_4)$.

So $(q_2, q_3, q_4) : q_1 = (q_2, q_3, q_4)$. By the above short exact sequence, we have $\text{pd}(S/I) = 4$. □

6. $\langle 2; 2 \rangle$ STRUCTURES

The first main result of this section is a structure theorem for the linear and quadric components of an unmixed ideal of height 3 of type $\langle 2; 2 \rangle$. It is worth noting that the results in [17] show that there are infinitely many such ideals; however, for the purposes of this paper we do not need a full structure theorem. Instead, we need only characterize the degree 2 part of such ideals.

Proposition 6.1. *Let L be a height 3 unmixed ideal primary to a multiplicity 2 prime $\mathfrak{p} = (x, y, q)$ (here x, y are linear forms and q is a quadric) with $\lambda(S_{\mathfrak{p}}/L_{\mathfrak{p}}) = 2$. Further assume that L contains a height 3 ideal I generated by quadrics. Then L has one of the following forms:*

- (1) $L = (x, y^2, q)$,
- (2) $L = (x^2, xy, y^2, q) + L'$, where L' is generated in degrees ≥ 3 ,
- (3) $L = (x^2, xy, y^2, ax+by, q) + L'$, where a, b are linear forms, $\text{ht}(x, y, a, b, q) = 5$ and L' is generated in degrees ≥ 3 ,
- (4) $L = (x^2, xy, y^2, ax + by, cx + dy, ad - bc + ex + fy = q)$, where a, b, c, d, e, f are linear forms, $\text{ht}(x, y, a, b) = \text{ht}(x, y, c, d) = 4$ and $ad - bc \notin (x, y)$.

Proof. First, notice that $q \in L$. Indeed, if not, every quadric in L is contained in (x, y) ; in particular this implies $\text{ht}(I) \leq 2$, which is a contradiction. Notice that the Hilbert function of L locally at \mathfrak{p} is $HF_{S_{\mathfrak{p}}/L_{\mathfrak{p}}} : 1, 1, 0, 0, \dots$. In particular, $\mathfrak{p}_{\mathfrak{p}}^2 \subseteq L_{\mathfrak{p}}$. Since L is \mathfrak{p} -primary, this yields $\mathfrak{p}^2 \subseteq L \subseteq \mathfrak{p}$. We conclude that $(x^2, xy, y^2, q) \subseteq L$.

Assume L also contains a linear form. Without loss of generality we may assume it is x . Then $(x, y^2, q) \subseteq L$. However, x, y^2, q is a complete intersection ideal contained in the unmixed ideal L and $e(S/(x, y^2, q)) = e(S/L) = 4$; hence $L = (x, y^2, q)$ by Lemma 2.3, and we are in case (1) of the statement.

We now assume L does not contain a linear form. Since $e(S/(x^2, xy, y^2, q)) = 6$, there must be at least an additional generator for L . If all such additional generators have degree at least three, then we are in case (2) of the statement. We may then assume L contains a quadric generator $q' \notin (x^2, xy, y^2, q)$. Since $q \in L$, without loss of generality we may assume q' is of the form $q' = ax + by$, where a, b are linear forms with $\text{ht}(x, y, a, b) = 4$. (Clearly we may assume that $\text{ht}(x, y, a, b) \geq 3$. If $\text{ht}(x, y, a, b) = 3$, then a, b share a common factor mod (x, y) , say $c \in \mathfrak{m}$. Writing $a = ca'$ and $b = cb'$ mod (x, y) , we have $ax + by = c(a'x + b'y)$ mod $(x, y)^2$. Since $(x, y)^2 \subseteq L$ and since q' is a minimal generator of L , it follows that c is a nonzerodivisor

on S/L . Since L is (x, y, q) -primary and c is a linear form, $c \in (x, y)$, which contradicts $\text{ht}(x, y, a, b) = 3$. Therefore $\text{ht}(x, y, a, b) = 4$.)

If $\text{ht}(x, y, a, b, q) = 5$ and there are no additional quadric generators of L , then we are in case (3). If $\text{ht}(x, y, a, b, q) = 4$, then we may write $q = fx + ey + da - cb$ for linear forms c, d, e, f . Then $b(cx + dy) = -x(fx + ey + da - cb) + f(x^2) + e(xy) + d(ax + by) \in L$. Since $b \notin (x, y, q)$ and L is (x, y, q) -primary, $(cx + dy) \in L$. If $\text{ht}(x, y, c, d) = 2$, then $q \in (x, y)$, a contradiction. If $\text{ht}(x, y, c, d) = 3$, then c, d share a common factor modulo (x, y) and again we have a linear form in L .

Hence, we may assume there is an additional quadric generator $q'' \notin (x^2, xy, y^2, q, q')$, where $q'' = cx + dy$ such that c, d are linear forms with $\text{ht}(x, y, c, d) = 4$. Since $HF_{S_p/L_p}(1) = 1$, the images of $ax + by$ and $cx + dy$ must be linearly dependent in S_p , that is $\Delta = ad - bc \in \mathfrak{p}_p \cap S = \mathfrak{p}$. Therefore, we can write $\Delta = ex + fy + \alpha q$ for linear forms e, f and $\alpha \in k$.

If $\alpha = 0$, then $\Delta \in (x, y)$. So $\Delta = 0$ (modulo (x, y)). By Lemma 4.1, $ax + by = \alpha(cx + dy)$ modulo $(x, y)^2$, which contradicts that $q'' = cx + dy \notin (x^2, xy, y^2, q, q' = ax + by)$. Hence $\alpha \neq 0$, and we may assume that $q = ad - bc + ex + fy$, for linear forms e, f .

Set $J = (x^2, xy, y^2, ax + by, cx + dy, ad - bc + ex + fy)$. By the following Lemma 6.2, J has multiplicity 4 and is Cohen-Macaulay; in particular, J is unmixed, $\text{ht}(J) = \text{ht}(L) = 3$ and $J \subset L$; hence $J = L$ by Lemma 2.3. \square

Lemma 6.2. *Let S be a polynomial ring, $a, b, c, d, e, f, x, y \in S_1$ and consider the ideal $J = (x^2, xy, y^2, ax + by, cx + dy, ad - bc + ex + fy)$. If $\text{ht}(x, y) = 2$ and $b, ad - bc \notin (x, y)$, then J is Cohen-Macaulay, $\text{ht}(J) = 3$ and $e(S/J) = 4$.*

Proof. We show that S/J has resolution

$$0 \longrightarrow S(-4)^3 \xrightarrow{\partial_3} S(-3)^8 \xrightarrow{\partial_2} S(-2)^6 \xrightarrow{\partial_1} S,$$

where

$$\begin{aligned}\partial_1 &= (x^2 \quad xy \quad y^2 \quad xa + yb \quad xc + yd \quad bc - ad - xe - yf), \\ \partial_2 &= \begin{pmatrix} -y & 0 & -a & 0 & -c & 0 & 0 & e \\ x & -y & -b & -a & -d & -c & e & f \\ 0 & x & 0 & -b & 0 & -d & f & 0 \\ 0 & 0 & x & y & 0 & 0 & -c & d \\ 0 & 0 & 0 & 0 & x & y & a & -b \\ 0 & 0 & 0 & 0 & 0 & 0 & y & x \end{pmatrix}, \text{ and} \\ \partial_3 &= \begin{pmatrix} a & c & -e \\ b & d & -f \\ -y & 0 & c \\ x & 0 & d \\ 0 & -y & -a \\ 0 & x & -b \\ 0 & 0 & x \\ 0 & 0 & -y \end{pmatrix}.\end{aligned}$$

The hypotheses imply $\text{ht}(I) = 3$. It is easy to check that $\partial_1, \partial_2, \partial_3$ define a complex and that $\text{ht}(I_1(\partial_1)) \geq 1$, $\text{ht}(I_5(\partial_2)) \geq 2$, and $\text{ht}(I_3(\partial_3)) \geq 3$. (To check the last statement, note that x^3, y^3 , and $b(ad - bc) + x(be - af) \in I_3(\partial_3)$.) Exactness follows from the Buchsbaum-Eisenbud acyclicity criteria (cf. [7, Theorem 20.9]). Hence, $\text{pd}(S/J) = 3$ and J is Cohen-Macaulay.

Finally, since S/J has a pure resolution of type $(2, 3, 4)$, we have $e(S/J) = \frac{2 \cdot 3 \cdot 4}{3!} = 4$ by [15, Theorem 1.2]. \square

Proposition 6.3. *If I is contained in an unmixed, height 3 ideal L of type $\langle 2; 2 \rangle$, then $\text{pd}(S/I) \leq 8$.*

Proof. Note that L is of one of the four types of ideals listed in Theorem 6.1. If L is as in case (2), then since all four quadrics of I can be expressed in terms of the regular sequence x, y, q , $\text{pd}(S/I) \leq 3$ by Lemma 2.21. If L is as in case (3), then all four quadrics of I can be expressed in terms of the regular sequence x, y, a, b, q , and $\text{pd}(S/I) \leq 5$ also by Lemma 2.21. If L is as in case (4), then all four quadric generators of I can be expressed in terms of at most 8 linear forms; hence $\text{pd}(S/I) \leq 8$.

Hence we may assume $L = (x, y^2, q)$, where x, y, q is a linear sequence of two linear forms and a quadric form as in case (1) of Theorem 6.1. If $e(S/I) = e(S/I^{un}) \geq 6$, the bound follows by Theorem 2.20. Hence we may assume $4 \leq e(S/I) \leq 5$; that is, either $I^{un} = (x, y^2, q)$, or $I^{un} = (x, y^2, q) \cap p'$, where $p' = (u, v, w)$ is a linear prime.

If $I^{un} = (x, y^2, q)$, then I^{un} is Cohen-Macaulay. It follows from Lemma 3.4 that $\text{pd}(S/I) \leq 4$.

Now assume $I^{un} = (x, y^2, q) \cap (u, v, w)$, where u, v, w are linear forms. If $x \in (u, v, w)$, then I^{un} contains a linear form and $\text{pd}(S/I) \leq 5$ by Lemma 3.3. If $x \notin (u, v, w)$, it follows from Lemma 3.5 that $y^2, q \in$

(u, v, w) . By Lemma 3.6, I^{un} is Cohen-Macaulay. Hence $\text{pd}(S/I) \leq 4$ by Lemma 3.4. \square

7. I^{un} HAS A PRIME COMPONENT OF MULTIPLICITY AT LEAST 2

In this section we handle the case where $I^{un} = \mathfrak{p} \cap J$, where \mathfrak{p} is a height three prime with $e(S/\mathfrak{p}) \geq 2$. We use the structure theorems on primes of small multiplicity from Section 2 as well as some technical results below.

We begin with the case when $e(S/\mathfrak{p}) = 5$.

Proposition 7.1. *If I is contained in a homogeneous height three prime ideal \mathfrak{p} with $e(S/\mathfrak{p}) = 5$, then $\text{pd}(S/I) \leq 5$.*

Proof. If $e(S/I) = e(S/I^{un}) \geq 6$, the bound follows by Theorem 2.20. Hence, the only case one has to consider is when $I^{un} = \mathfrak{p}$. If \mathfrak{p} is degenerate, by Lemma 3.3 we have $\text{pd}(S/I) \leq 5$. Hence, we may assume \mathfrak{p} is non-degenerate. Then, by Theorem 2.18, the only prime \mathfrak{p} that contains 4 linearly independent quadrics is the ideal in case (2). By Lemma 3.7,

$$\text{pd}(S/I) = \max \{5, \text{pd}(\text{Ker}(d_4^*)) + 2\},$$

where d_4 is the fourth differential map in the minimal free resolution of S/\mathfrak{p} . From the Betti table we see that d_4 is represented by a 5×1 matrix of linear forms, say ℓ_1, \dots, ℓ_5 . Hence

$$\text{pd}(\text{Ker}(d_4^*)) = \text{pd}(\text{Ker}(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)) = \text{pd}(S/(\ell_1, \dots, \ell_5)) - 2 \leq 3.$$

Hence $\text{pd}(S/I) \leq 5$. \square

Now we handle the case where I^{un} has a prime component of multiplicity 4.

Proposition 7.2. *If I is contained in a homogeneous prime ideal \mathfrak{p} with $e(S/\mathfrak{p}) = 4$, then $\text{pd}(S/I) \leq 8$.*

Proof. Again, if $e(S/I) = e(S/I^{un}) \geq 6$, the bound follows by Theorem 2.20. Hence, the only cases one has to consider are $I^{un} = \mathfrak{p}$ (when $e(S/I) = 4$) and $I^{un} = \mathfrak{p} \cap \mathfrak{p}'$, where \mathfrak{p}' is a linear prime (when $e(S/I) = 5$). Note that we may assume that \mathfrak{p} is as in cases (2), (3) or (4) of Theorem 2.17, as cases (1) and (5) cannot contain a height 3 ideal minimally generated by 4 quadrics.

First, assume $I^{un} = \mathfrak{p}$. Since \mathfrak{p} is Cohen-Macaulay for every possibility, $\text{pd}(S/I) \leq 4$ by Lemma 2.6.

We may then assume $e(S/I) = 5$ and $I^{un} = \mathfrak{p} \cap \mathfrak{p}'$, where $\mathfrak{p}' = (a, b, c)$ is a linear prime. If \mathfrak{p} is degenerate (case (2)), then $\mathfrak{p} = (x, q, q')$ for quadrics q, q' . By two applications of Lemma 3.5, we may assume that $q, q' \in (a, b, c)$. Since $x \notin (a, b, c)$, we apply Lemma 3.6 to get $\text{pd}(S/I) \leq 4$.

Finally, if \mathfrak{p} is non-degenerate (cases (3) and (4)), then the generators of I can be expressed in terms of at most 8 linear forms; hence, $\text{pd}(S/I) \leq 8$. \square

We now prove a similar result for primes of multiplicity 3.

Proposition 7.3. *If I is contained in a homogeneous prime ideal \mathfrak{p} with $e(S/\mathfrak{p}) = 3$, then $\text{pd}(S/I) \leq 9$.*

Proof. If $e(S/I) = e(S/I^{un}) \geq 6$, the bound follows by Theorem 2.20. Hence, one needs only consider the cases where $I^{un} = \mathfrak{p}$, $I^{un} \subseteq \mathfrak{p} \cap \mathfrak{p}'$, where \mathfrak{p}' is a prime of multiplicity 1 or 2, or $I^{un} \subseteq \mathfrak{p} \cap L$, where $e(S/L) = 2$ and L is primary to a linear prime.

By Theorem 5.1, in this latter case we have $\text{pd}(S/I) \leq 9$. Also, if $I^{un} = \mathfrak{p}$, by Lemma 2.14, \mathfrak{p} is degenerate. It then follows from Lemma 3.3 that $\text{pd}(S/I) \leq 5$.

So we only need to consider the cases where $I^{un} \subseteq \mathfrak{p} \cap \mathfrak{p}'$, where \mathfrak{p}' is a prime of multiplicity 1 or 2. Then $\mathfrak{p} = (x) + I_2(\mathbf{M})$, where \mathbf{M} is a matrix as in case (2) of Theorem 2.16. (Case (1) of Theorem 2.16 cannot occur, since then $I \subset (x, y)$, contradicting $\text{ht}(I) = 3$.)

First, assume $I^{un} \subseteq \mathfrak{p} \cap (a, b, c)$, where a, b, c are linear forms. If $x \in (a, b, c)$, then x is a linear form in I^{un} and $\text{pd}(S/I) \leq 5$ by Lemma 3.3. If $x \notin (a, b, c)$, then the quadric generators of I are completely expressible in terms of x, a, b, c along with the 6 entries of \mathbf{M} . All 10 variables are linearly independent, then $(I^{un})_2 \subset (xa, xb, xc)$, again contradicting $\text{ht}(I) = 3$. Hence the generators of I are expressible in terms of at most 9 linear forms, giving $\text{pd}(S/I) \leq 9$.

Finally, consider the case where $I^{un} \subseteq \mathfrak{p} \cap \mathfrak{p}'$, where $e(S/\mathfrak{p}') = 2$. Let $\Delta_1, \Delta_2, \Delta_3$ be the three 2×2 minors of \mathbf{M} . By Corollary 2.15, $\mathfrak{p}' = (a, b, q)$, where a, b are linear forms and q is a quadric. By Lemma 3.3, we may assume $x \notin \mathfrak{p}'$. By Lemma 3.5, there exists a quadric q' such that $(a, b, q) = (a, b, q')$ and $q' \in \mathfrak{p}$. Writing $q' = xy + \sum_i \alpha_i \Delta_i$, where y is a linear form and $\alpha_i \in k$ for $i = 1, 2, 3$. We now have that all the generators of I are expressible in terms of the 10 linear forms x, y, a, b along with the 6 entries of \mathbf{M} . If all 10 forms are linearly independent, then

$$\mathfrak{p} \cap \mathfrak{p}' = (ax, bx, q', a\Delta_1, a\Delta_2, a\Delta_3, b\Delta_1, b\Delta_2, b\Delta_3),$$

which cannot contain a height three ideal generated by quadrics. Hence the generators of I are expressible in terms of at most 9 linear forms and $\text{pd}(S/I) \leq 9$. \square

To prove the corresponding result for multiplicity 2 primes, we need a couple of technical results.

Lemma 7.4. *Assume I is contained in the intersection of two distinct linear primes. Then either $\text{pd}(S/I) \leq 9$ or there exist linear forms a, x, y, z such that $I \subseteq (x, y, az)$.*

Proof. By assumption, there exist linear forms a, b, c and x, y, z such that $I \subseteq (x, y, z) \cap (a, b, c)$. If $\text{ht}(x, y, z, a, b, c) = 6$, then every quadric of I can be written in terms of this 6 linear forms, whence $\text{pd}(S/I) \leq 6$.

Suppose $\text{ht}(x, y, z, a, b, c) = 5$. Without loss of generality, we may assume $x = c$, so that $I \subseteq (x, y, z) \cap (a, b, x) = (x, ay, by, az, bz)$. For every $i = 1, \dots, 4$ let ℓ_i be a linear form such that $q_i = \ell_i x + q_i$ for some

$q_i \in (ay, by, az, bz)$. Then, quadric element of I can be written in terms of the 9 linear forms $a, b, x, y, z, \ell_1, \dots, \ell_4$, giving $\text{pd}(S/I) \leq 9$.

Finally, if $\text{ht}(x, y, z, a, b, c) = 4$, without loss of generality we may assume $x = b$ and $y = c$, so that $I \subseteq (x, y, z) \cap (x, y, c) = (x, y, az)$. \square

Lemma 7.5. *Suppose x, y are linear forms, q is a quadric so that $J = (x, y, q)$ is a complete intersection. Further suppose that $I \subseteq J \cap \mathfrak{p}$, where $\mathfrak{p} = (a, b, c)$ is a linear prime of height 3 not containing J . Then, either $\text{pd}(S/I) \leq 8$ or the linear forms x, y, a, b are independent and we may take quadric $q \in (a, b)$ so that $I \subseteq (x, ay, by, q)$.*

Proof. Suppose $\text{ht}(x, y, a, b, c) = 3$. In this case, we may take $a = x$ and $b = y$, so we are intersecting (x, y, q) with (x, y, c) . Then, by Lemma 3.5 we may assume $q \in (x, y, c)$, showing that $J \subseteq \mathfrak{p}$. This contradicts that $J \not\subseteq \mathfrak{p}$.

If $\text{ht}(x, y, a, b, c) = 5$, by Lemma 3.5 we may assume that $q \in (a, b, c)$. We have $(x, y, q) \cap (a, b, c) = (q) + [(x, y) \cap (a, b, c)] = (q, ax, ay, bx, by, cx, cy)$. Write $q = \ell_1 a + \ell_2 b + \ell_3 c$. Then, every quadric in this intersection is written in terms of the 8 linear forms $a, b, c, x, y, \ell_1, \ell_2, \ell_3$. This gives the bound $\text{pd}(S/I) \leq 8$.

Finally, if $\text{ht}(x, y, a, b, c) = 4$, one may take $c = x$. By Lemma 3.5 we may assume that $q \in (a, b, x)$. Moreover, since $x \in (x, y, q) \cap (a, b, x)$, we may assume that $q \in (a, b)$. \square

Next, we prove that $\text{pd}(S/I) \leq 9$ when I is contained in the intersection of two special complete intersections.

Proposition 7.6. *Let a, b, x, y be linear forms, and let q and r be quadrics. If $I \subseteq (a, b, r) \cap (x, y, q)$ and neither of (a, b, r) and (x, y, q) are contained in the other, then $\text{pd}(S/I) \leq 9$.*

Proof. Since $\text{ht}(I) = 3$, we have $\text{ht}(a, b, x, y) \geq 2$. Suppose $\text{ht}(a, b, x, y) = 2$. We can assume $a = x$ and $b = y$. By Lemma 3.5, we may assume $q \in (a, b, r)$, giving that $(a, b, q) \subseteq (a, b, r)$. This is a contradiction.

Now, assume $\text{ht}(a, b, x, y) = 3$. We may assume that $x = b$. By Lemma 3.5, we may assume that $q \in (a, x, r)$ and $r \in (x, y, q)$. Further, we may assume that q is of the form $wy + \alpha r$ for some linear form w and $\alpha \in k$, and we may write $r = za + \beta q$ for a linear form z and $\beta \in k$. If $\alpha = \beta^{-1}$, then it follows that $wy = -\alpha qz$. Since $\text{ht}(a, x, y) = 3$, we may assume $a = w$. But then $(a, b, q) = (a, b, r)$, a contradiction. If not, then it follows that $q = (1 - \alpha\beta)^{-1}(wy - \alpha za)$, and $r = (1 - \alpha\beta)^{-1}(za + \beta wy)$. We then have $(x, a, r) \cap (x, y, q) = (x, a, wy) \cap (x, y, za) = (x, wy, za, ay)$. Therefore, we may write $q_i = \ell_i x + \gamma_i wy + \delta_i za + \epsilon_i ay$, $i = 1$ to 4 , where ℓ_1, \dots, ℓ_4 are linear forms and $\gamma_i, \delta_i, \epsilon_i \in k$ for all i . Hence the generators of I can be written only in terms of $x, a, y, w, z, \ell_1, \dots, \ell_4$, that is, in terms of at most 9 linear forms. This gives the desired bound of 9.

Finally, consider the case where $\text{ht}(a, b, x, y) = 4$. By Lemma 3.5 we may assume $q \in (a, b, r)$ and $r \in (x, y, q)$. We then obtain

$$(a, b, r) \cap (x, y, q) = (q, r) + [(a, b) \cap (x, y)] = (q, r, ax, ay, bx, by).$$

Similar to the above, we may write $q = \ell_1 a + \ell_2 b + \alpha r$ and $r = \ell_3 x + \ell_4 y + \beta q$ for linear forms ℓ_1, \dots, ℓ_4 and $\alpha, \beta \in k$. If $\alpha^{-1} = \beta$, we obtain $0 = \ell_1 a + \ell_2 b + \alpha \ell_3 x + \alpha \ell_4 y$. Since x, y, a, b is a regular sequence, it follows that $\ell_1, \dots, \ell_4 \in (x, y, a, b)$. It follows that $I \subseteq (q, r, ax, ay, bx, by) \subseteq (q) + (a, b, x, y)^2$. If a, b, x, y, q is a regular sequence, we have that $\text{pd}(S/I) \leq 5$ by Lemma 2.21. If not, then $q = \ell'_1 a + \ell'_2 b + \ell'_3 x + \ell'_4 y$ for linear forms ℓ'_1, \dots, ℓ'_4 , and the generators of I can be written in terms of at most 8 linear forms; that is $\text{pd}(S/I) \leq 8$. Finally if $\alpha^{-1} \neq \beta$, then as before, q and r can each be written in terms of $a, b, x, y, \ell_1, \dots, \ell_4$ only. The same then holds for the generators of I and $\text{pd}(S/I) \leq 8$. \square

We are now able to prove the bound on $\text{pd}(S/I)$ whenever I is contained in a prime of multiplicity two.

Proposition 7.7. *If I is contained in a homogeneous prime ideal \mathfrak{p} with $e(S/\mathfrak{p}) = 2$, then $\text{pd}(S/I) \leq 9$.*

Proof. If $e(S/I) = e(S/I^{un}) \geq 6$, the bound follows by Theorem 2.20. Hence we may assume $2 \leq e(S/I) \leq 5$. Then, either $I^{un} = \mathfrak{p}$, or $I^{un} \subseteq \mathfrak{p} \cap \mathfrak{p}'$, where \mathfrak{p}' is either a prime of multiplicity 1, 2 or 3, or $I^{un} \subseteq \mathfrak{p} \cap L$, where $e(S/L) = 2$ or 3 and L is primary to a linear prime. By Proposition 5.1, we already know that in these latter two cases one has $\text{pd}(S/I) \leq 9$. Also, the case $I^{un} \subseteq \mathfrak{p} \cap \mathfrak{p}'$, where \mathfrak{p}' is a prime of multiplicity 3, has been proved in Proposition 7.3.

Next, consider the case where $I^{un} = \mathfrak{p}$. By Corollary 2.15, \mathfrak{p} is a complete intersection ideal, hence $\text{pd}(S/I) \leq 4$ by Lemma 3.4.

We need only to consider the case $I^{un} \subseteq \mathfrak{p} \cap \mathfrak{p}'$, where \mathfrak{p}' is either a prime of multiplicity 1 or 2. If $e(S/\mathfrak{p}') = 2$, by Corollary 2.15, there exist linear forms x, y, a, b , and quadrics q, r so that $\mathfrak{p} = (x, y, q)$ and $\mathfrak{p}' = (a, b, r)$. Then $\text{pd}(S/I) \leq 9$ by Proposition 7.6.

Finally, we assume $I^{un} \subseteq \mathfrak{p} \cap \mathfrak{p}'$, where \mathfrak{p}' is a linear prime. First, assume $I^{un} \subsetneq \mathfrak{p} \cap \mathfrak{p}'$. Then, the results we proved above show that $\text{pd}(S/I) \leq 9$ unless $I^{un} \subseteq \mathfrak{p} \cap \mathfrak{p}' \cap \mathfrak{p}''$, where also \mathfrak{p}'' is a linear prime. By Lemma 7.4, either $\text{pd}(S/I) \leq 9$, or $\mathfrak{p}' \cap \mathfrak{p}'' = (a, b, r)$ for a quadric r . By Corollary 2.15, $\mathfrak{p} = (x, y, q)$ for linear forms x, y and a quadric q . Since $I \subseteq (x, y, q) \cap (a, b, r)$, Proposition 7.6 gives the desired bound in this situation.

We may then assume that $I^{un} = \mathfrak{p} \cap \mathfrak{p}'$. Write $\mathfrak{p} = (x, y, q)$ and $\mathfrak{p}' = (a, b, c)$, where x, y, a, b, c are linear forms and q is a quadric. If $\text{ht}(a, b, c, x, y) < 5$, then we may assume $x = c$, which is a linear form contained in I^{un} . Now, Lemma 3.3 gives $\text{pd}(S/I) \leq 5$. Hence, we may assume $\text{ht}(a, b, c, x, y) = 5$. By Lemma 3.5, we may assume $q \in (a, b, c)$, showing that $\mathfrak{p} \cap \mathfrak{p}' = (q) + [(x, y) \cap (a, b, c)] = (q, ax, ay, bx, by, cx, cy)$, where the last equality follows because $\text{ht}(a, b, c, x, y) = 5$. Writing $q = \ell_1 a + \ell_2 b + \ell_3 c$ for linear forms ℓ_1, ℓ_2, ℓ_3 , shows that every quadric of I can be written only in terms of $a, b, c, x, y, \ell_1, \ell_2, \ell_3$, yielding $\text{pd}(S/I) \leq 8$. \square

8. I^{un} IS AN INTERSECTION OF LINEAR PRIMES OF HEIGHT 3

The only case not covered by the previous sections is when the unmixed part of I is the intersection of several linear primes. The following result handles this last case.

Proposition 8.1. *If I^{un} is the intersection of at least two distinct linear primes, then $\text{pd}(S/I) \leq 9$.*

Proof. If $e(S/I) = e(S/I^{un}) \geq 6$, the bound follows by Theorem 2.20. Hence we may assume $I^{un} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \dots \cap \mathfrak{p}_n$ where $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are linear primes, and $2 \leq n \leq 5$.

If $n \geq 4$, we have $I^{un} \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3 \cap \mathfrak{p}_4$. By Lemma 7.4, we may assume that either $\text{pd}(S/I) \leq 9$ or that $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (a, b, r)$ for linear forms a, b and a quadric r . Similarly, we may assume $\mathfrak{p}_3 \cap \mathfrak{p}_4 = (x, y, q)$ for linear forms x, y and a quadric q . Since $I \subseteq (x, y, q) \cap (a, b, r)$, Proposition 7.6 proves $\text{pd}(S/I) \leq 9$.

Hence, we only need to prove $\text{pd}(S/I) \leq 9$, when $n = 2, 3$. When $n = 2$, we have $I^{un} = (a, b, c) \cap (x, y, z)$ for linear forms x, y, z, a, b, c . If $\text{ht}(a, b, c, x, y, z) < 6$, then there exists a linear form in I^{un} , giving $\text{pd}(S/I) \leq 5$ by Lemma 3.3. On the other hand, if $\text{ht}(a, b, c, x, y, z) = 6$, we have $I^{un} = (ax, ay, az, bx, by, bz, cx, cy, cz)$, showing that every quadric of I can be written in terms of at most 6 linear forms. This gives $\text{pd}(S/I) \leq 6$.

Therefore, the only case left is when $I^{un} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3$. Write $\mathfrak{p}_1 = (x, y, z)$. By Lemma 7.4, we may assume $\mathfrak{p}_2 \cap \mathfrak{p}_3 = (a, b, q)$ for linear forms a, b and a quadric q . By Lemma 7.5 we may assume $q \in (x, y, z)$. If $\text{ht}(a, b, x, y, z) < 5$, then there exists a linear form in I^{un} , giving $\text{pd}(S/I) \leq 5$ by Lemma 3.3. On the other hand, if $\text{ht}(a, b, x, y, z) = 5$, then

$$I^{un} = (x, y, z) \cap (a, b, q) = (q) + [(x, y, z) \cap (a, b)] = (q, ax, ay, az, bx, by, bz).$$

Writing $q = \ell_1 x + \ell_2 y + \ell_3 z$ for linear forms ℓ_1, ℓ_2, ℓ_3 , shows that every quadric in I can be written in terms of at most 8 linear forms, namely, $a, b, x, y, z, \ell_1, \ell_2, \ell_3$. Hence $\text{pd}(S/I) \leq 8$. \square

9. PROOF OF THE MAIN RESULT

In this section we combine our previous results to prove our main result, which we state again here.

Theorem 9.1. *Let S be a polynomial ring over a field k , and let $I = (q_1, q_2, q_3, q_4)$ be an ideal of S generated by 4 homogeneous polynomials of degree 2. Then $\text{pd}(S/I) \leq 9$.*

Proof. By extending the base field, we may assume that k is algebraically closed. First, assume $\text{ht } I = 1$, then there exist linear forms x, y_1, \dots, y_4 with $q_i = xy_i$ for every $i = 1, \dots, 4$. The short exact sequence

$$0 \longrightarrow S/(y_1, \dots, y_4) \xrightarrow{\cdot x} S/I \longrightarrow S/(x) \longrightarrow 0$$

together with Depth Lemma gives $\text{pd}(S/I) \leq 4$.

The case where $\text{ht}(I) = 2$ has been considered in [16], where we proved that, in this setting, $\text{pd}(S/I) \leq 6$.

If $\text{ht}(I) = 3$, then $e(S/I) \leq 6$ by Theorem 2.20. Moreover, I is Cohen-Macaulay if $e(S/I) = 6$. Hence we may assume $e(S/I) \leq 5$. By considering the primary components of I^{un} we have the following possibilities:

- (1) I^{un} has a component of type $\langle 2; 2 \rangle$.
- (2) I^{un} has a prime component of multiplicity 2, 3, 4 or 5.
- (3) I^{un} has a component of type $\langle 1; e \rangle$ with $e \geq 2$.
- (4) I^{un} is precisely the intersection of at most 5 linear primes.

In case (1), we have $\text{pd}(S/I) \leq 8$ by Proposition 6.3. In case (2) we have $\text{pd}(S/I) \leq 9$ by Propositions 7.1, 7.2, 7.3 and 7.7. In case (3), we have $\text{pd}(S/I) \leq 9$ by Proposition 5.1. In case (4) we have $\text{pd}(S/I) \leq 9$ by Proposition 8.1. For the reader's convenience we summarize all the cases in Table 1, in the next page.

Finally, if $\text{ht}(I) = 4$, then I is a complete intersection ideal giving $\text{pd}(S/I) = 4$. \square

We remark that there are canonical examples of ideals I_2 and I_3 generated by 4 quadrics of heights 2 and 3, respectively, and with $\text{pd}(S/I_2) = \text{pd}(S/I_3) = 6$. Namely, we can take $I_2 = (x^2, y^2, ax + by, cx + dy)$ and $I_3 = (x^2, y^2, z^2, ax + by + cz)$ in the polynomial ring $S = k[a, b, c, d, x, y, z]$. These correspond to the examples $I_{2,2,2}$ and $I_{3,1,2}$ from [23]. We expect that 6 is the maximum attainable projective dimension for any ideal generated by 4 quadrics. More generally, we previously posed the following question:

Question 9.2 ([16, Question 6.2]). *Let R be a polynomial ring and let I be an ideal of R generated by n quadrics and having $\text{ht } I = h$. Is it true that $\text{pd}(R/I) \leq h(n - h + 1)$?*

There are examples in [23] achieving this bound for all possible integers h and n . Note that if the question has a positive answer, it would give a bound on the projective dimension of ideals generated by n quadrics that is quadratic in n – much smaller than the known bounds of Ananyan-Hochster.

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$\mathbf{e}(S/I)$	$\langle \underline{e}; \underline{\lambda} \rangle$ for I^{un}	Bound for $\text{pd}(S/I)$	Justification
1	$\langle 1; 1 \rangle$	4	Proposition 3.2
2	$\langle 2; 1 \rangle$	4	Lemma 3.4
	$\langle 1; 2 \rangle$	9	Proposition 5.1
	$\langle 1, 1; 1, 1 \rangle$	5	Proposition 8.1
3	$\langle 3; 1 \rangle$	5	Lemma 3.3
	$\langle 1; 3 \rangle$	9	Proposition 5.1
	$\langle 1, 2; 1, 1 \rangle$	8	Proposition 7.7
	$\langle 1, 1; 1, 2 \rangle$	9	Proposition 5.1
	$\langle 1, 1, 1; 1, 1, 1 \rangle$	9	Proposition 8.1
4	$\langle 4; 1 \rangle$	4	Lemma 3.4
	$\langle 2; 2 \rangle$	8	Proposition 6.3
	$\langle 1; 4 \rangle$	9	Proposition 5.1
	$\langle 1, 3; 1, 1 \rangle$	9	Proposition 7.3
	$\langle 1, 1; 1, 3 \rangle$	9	Proposition 5.1
	$\langle 2, 2; 1, 1 \rangle$	9	Proposition 7.7
	$\langle 1, 2; 2, 1 \rangle$	9	Proposition 5.1
	$\langle 1, 1; 2, 2 \rangle$	9	Proposition 5.1
	$\langle 1, 1, 2; 1, 1, 1 \rangle$	9	Proposition 8.1
	$\langle 1, 1, 1; 1, 1, 2 \rangle$	9	Proposition 5.1
	$\langle 1, 1, 1, 1; 1, 1, 1, 1 \rangle$	9	Proposition 8.1
5	$\langle 5; 1 \rangle$	5	Proposition 7.1
	$\langle 1; 5 \rangle$	9	Proposition 5.1
	$\langle 1, 4; 1, 1 \rangle$	8	Proposition 7.2
	$\langle 1, 2; 1, 2 \rangle$	8	Proposition 6.3
	$\langle 1, 1; 1, 4 \rangle$	9	Proposition 5.1
	$\langle 2, 3; 1, 1 \rangle$	9	Proposition 7.3
	$\langle 2, 1; 1, 3 \rangle$	9	Proposition 5.1
	$\langle 3, 1; 1, 2 \rangle$	9	Proposition 5.1
	$\langle 1, 1; 2, 3 \rangle$	9	Proposition 5.1
	$\langle 1, 1, 3; 1, 1, 1 \rangle$	9	Proposition 7.3
	$\langle 1, 1, 1; 1, 1, 3 \rangle$	9	Proposition 5.1
	$\langle 1, 2, 2; 1, 1, 1 \rangle$	9	Proposition 7.7
	$\langle 1, 1, 2; 1, 2, 1 \rangle$	9	Proposition 5.1
	$\langle 1, 1, 1; 1, 2, 2 \rangle$	9	Proposition 5.1
	$\langle 1, 1, 1, 2; 1, 1, 1, 1 \rangle$	9	Proposition 7.7
	$\langle 1, 1, 1, 1; 1, 1, 1, 2 \rangle$	9	Proposition 5.1
	$\langle 1, 1, 1, 1, 1; 1, 1, 1, 1, 1 \rangle$	9	Proposition 8.1
6	any	3	Theorem 2.20

TABLE 1. Bounds on $\text{pd}(S/I)$ where I is a height 3 ideal generated by 4 quadrics

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UNIVERSITY OF VIRGINIA, DEPARTMENT OF MATHEMATICS, 141 CABELL DRIVE, KERCHOF HALL, P.O. BOX 400137, CHARLOTTESVILLE, VA 22904-4137

E-mail address: `huneke@virginia.edu`

UNIVERSITY OF CALIFORNIA RIVERSIDE, DEPARTMENT OF MATHEMATICS, 900 UNIVERSITY AVE., RIVERSIDE, CA 92521

E-mail address: `mantero@math.ucr.edu`

RIDER UNIVERSITY, DEPARTMENT OF MATHEMATICS, 2083 LAWRENCEVILLE ROAD, LAWRENCEVILLE, NJ 08648

E-mail address: `jmccullough@rider.edu`

UNIVERSITY OF NEBRASKA, DEPARTMENT OF MATHEMATICS, 203 AVERY HALL, LINCOLN, NE 68588

E-mail address: `aseceleanu2@math.unl.edu`